

# Boundness of Laplacian eigenfunctions on manifolds of infinite volume

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## Abstract

In this work we consider complete Riemannian manifolds in which there exists some isoperimetric function satisfying some integrability condition. For such a manifold  $M$ , we prove that if a  $\lambda$ -eigenfunction  $u$  of the Laplacian in  $M$  belongs to  $L^p(M)$  for some  $p \geq 2$ , then  $u$  is bounded and  $\|u\|_\infty \leq C\|u\|_p$  for some constant  $C$  that depends only on  $p$ ,  $\lambda$  and  $M$ . This estimate also holds for  $\lambda$ -eigenfunctions in a regular domain  $U \subset M$ . These results can be applied for any Hadamard manifold  $M$ .

## 1 Introduction

Let  $M$  be a complete Riemannian manifold. Given an open subset  $U \subset M$  of class  $C^2$  and  $\lambda \in \mathbb{R}$ , we call  $\lambda$ -eigenfunction of  $U$  any nontrivial  $u \in C^2(U)$  that satisfies

$$\Delta u + \lambda u = 0 \text{ in } U. \quad (1)$$

If in addition  $U$  has nonempty boundary, we require that  $u$  vanishes on  $\partial U$ .

When  $M$  is compact, according to the spectral theory for elliptic operators, the set of  $\lambda$  for which (1) has a nontrivial classical solution is formed by the terms of an increasing unbounded sequence  $(\lambda_k)$ . They are called the eigenvalues of  $-\Delta$  and are the only elements of its spectrum. If  $M$  is non-compact, the situation is more delicate since the spectrum of  $-\Delta$  can contain elements that are not eigenvalues. Furthermore, an  $\lambda$ -eigenfunction may not

belong to  $L^2(U)$  or even to  $L^\infty(U)$ . This raises the questions whether an  $\lambda$ -eigenfunction is in some  $L^q(U)$  and whether this implies its boundedness.

In this setting, A. Cianchi and V.G. Maz'ya [4] investigate bounds for eigenfunctions in noncompact manifolds of finite measure. They considered a slightly different eigenvalue problem, which coincides with (1) in the case of empty boundary. Under assumptions on the isoperimetric profile of  $M$ , the authors obtained

$$\|u\|_{L^\infty} \leq C\|u\|_{L^2} \quad (2)$$

for any  $u$   $\lambda$ -eigenfunction of  $M$ , where  $C > 0$  is a constant that depends only on the isoperimetric profile and on  $\lambda$ . Furthermore, their result is sharp in the sense that if  $H$  is a suitable isoperimetric profile that does not have the assumptions above mentioned, then there is a manifold  $M$  with isoperimetric profile close to  $H$  for small values of  $s$ , that admits an eigenfunction  $u \in L^2(M) \setminus L^\infty(M)$ .

Following these ideas, a natural question is under which geometric constraints the inequality (2) holds for manifolds of infinite measure. Our main result, Theorem 2.2, answers this question. Under the same assumptions on the isoperimetric profile considered in [4],  $\|u\|_{L^\infty} \leq C\|u\|_{L^p}$  holds for any  $\lambda$ -eigenfunction in  $L^p$ ,  $p \geq 2$ .

As a particular case of our result, we obtain that if  $M$  is a Hadamard manifold, any  $\lambda$ -eigenfunction that belongs to some  $L^p(M)$  must be bounded. For instance, the hyperbolic space  $\mathbb{H}^n(-c^2)$  does not admit eigenfunctions in  $L^2$ , but it has  $\lambda$ -eigenfunctions in  $L^p$  for  $p > 2$ . Any of these is bounded according to the main result of this work.

## 2 Main result

In this section, we state and prove our main Theorem. The proof presented here follows ideas from a norm estimate for eigenfunctions of some elliptic operators in domains of  $\mathbb{R}^n$  presented in [1]. In order to adapt these ideas, we need an associated isoperimetric function (a. i. f.), which arises when one changes from Euclidean space to Riemannian manifolds.

**Definition 2.1.** *Consider  $M$  a complete Riemannian manifold. An isoperimetric function on  $M$  is a function  $H : [0, \text{Vol}(M)] \rightarrow \mathbb{R}$  that satisfies*

$$H(\text{Vol}(\Omega)) \leq \text{Vol}(\partial\Omega) \quad \forall \Omega \subset\subset M. \quad (3)$$

If  $\text{Vol}(M) = \infty$ ,  $H$  is defined in  $\mathbb{R}_+$ .

Each isoperimetric function on  $M$  defines an associated isoperimetric function (a.i.f.) by

$$H_a(t) := \int_0^t \frac{s}{H(s)^2} ds, \quad t \in [0, \text{Vol}(M)],$$

if the integral converges.

**Theorem 2.2.** *Let  $M$  be a complete Riemannian manifold with well-defined a.i.f.  $H_a$ ,  $U \subset M$  a domain, possibly unbounded,  $\lambda > 0$  and  $p \geq 2$ . There exists a constant  $C = C(\lambda, p, H)$  such that for all nontrivial solution  $w \in W^{1,p}(U)$  of*

$$\begin{cases} -\Delta v = \lambda v & \text{in } U \\ v = 0 & \text{on } \partial U \end{cases} \quad (4)$$

it holds that

$$\|w\|_\infty \leq C \|w\|_p.$$

## 2.1 Lemmata

Until the end of this section,  $M$  will be endowed with an isoperimetric function  $H$  that has a well defined a.i.f.  $H_a$ .

**Lemma 2.3.** *Let  $\Omega \subset M$  be a domain with finite measure and  $u$  be a solution of  $-\Delta u = c$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ . Then*

$$\sup u \leq c H_a(|\Omega|).$$

*Proof.* We first consider the case that  $\Omega$  is bounded.

Let  $u$  be a solution of  $-\Delta u = c$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ . Consider  $\mu$  the distribution function of  $u$ , defined as  $\mu(t) = |\{x \in \Omega \mid |u(x)| > t\}|$ . It is known that for almost all  $t$

$$\mu'(t) = - \int_{\{u=t\}} \frac{1}{|\nabla u|} da_t. \quad (5)$$

For all  $t > 0$ , the set  $\{|u(x)| > t\}$  is at a positive distance from the boundary  $\partial\Omega$ . It is compactly contained in  $\Omega$ , has boundary  $\{u = t\}$  with

inner normal vector  $\frac{\nabla u}{|\nabla u|}$ , well defined for almost all  $t > 0$  by Sard's Theorem. If the inner normal vector is well defined for  $t$ , we apply the Divergence Theorem on the differential equation, obtaining

$$c\mu(t) = \int_{\{u>t\}} c \, dx = c \int_{\{u=t\}} |\nabla u| da_t. \quad (6)$$

Applying the isoperimetric function  $H$  in  $|\Omega_t|$ , using expression (5) for the derivative of  $\mu$  and Cauchy-Schwarz inequality, we obtain

$$H(\mu(t)) \leq |\{u = t\}| \leq -\mu'(t)^{1/2} \left( \int_{\{u=t\}} |\nabla u| da_t \right)^{1/2}.$$

Hence, for almost all  $t > 0$ ,

$$H^2(\mu(t)) \leq -\mu'(t)c\mu(t) \text{ and } 1 \leq \frac{c\mu(t)(-\mu'(t))}{H^2(\mu(t))} = -c \frac{d}{dt} H_a(\mu(t)).$$

Integrating the above inequality in  $[0, \sup u]$ ,

$$\sup u \leq -c (H_a(\mu(\sup u)) - H_a(|\Omega|)) = cH_a(|\Omega|),$$

the proof is complete for bounded  $\Omega$ .

To the general case, let  $u_n$  be the solution to the Dirichlet problem in  $\Omega_n$ , where  $(\Omega_n)$  is an increasing sequence of bounded sets such that  $\Omega = \cup \Omega_n$ . From the first case,  $(u_n)$  is uniformly bounded by  $cH_a(|\Omega|)$  and, from the maximum principle, is a increasing sequence. Hence  $u_n$  converges to a solution  $u$  that is bounded by  $cH_a(|\Omega|)$ , proving the result.  $\square$

**Proposition 2.4.** *Let  $\Omega \subset M$  be a bounded domain and  $w \in C^2(\Omega) \cap C^0(\bar{\Omega})$  be a nontrivial solution of*

$$\begin{cases} -\Delta v = \lambda v \text{ in } \Omega \\ v = \gamma \text{ on } \partial\Omega \end{cases}$$

for some  $0 < \lambda \leq \lambda_M$  and  $\gamma \in \mathbb{R}$ . Then

$$\frac{1}{2\lambda} \leq \frac{\|w\|_\infty}{\|w\|_\infty - |\gamma|} H_a \left( \left[ \frac{2\|w\|_p}{\|w\|_\infty + |\gamma|} \right]^p \right).$$

*Proof.* Denote by  $K = \|w\|_\infty$  and assume that  $\max w = \max |w|$ . We can suppose that since  $w$  cannot change sign, otherwise  $\lambda \geq \lambda_1(\Omega) > \lambda_M$ . Let

$$\tilde{\Omega} = \left\{ x \in \Omega \mid |w(x)| > \frac{K + |\gamma|}{2} \right\}.$$

Then,

$$\|w\|_p^p = \int_{\Omega} |w|^p dx \geq \int_{\tilde{\Omega}} |w|^p dx \geq \left( \frac{K + |\gamma|}{2} \right)^p |\tilde{\Omega}| \quad (7)$$

On the other hand,

$$-\Delta w = \lambda w \leq \lambda K.$$

By the comparison principle,  $w \leq u$  in  $\tilde{\Omega}$  where  $u$  is solution of

$$\begin{cases} -\Delta v = \lambda K & \text{in } \tilde{\Omega} \\ v = \frac{K + |\gamma|}{2} & \text{on } \partial\tilde{\Omega} \end{cases}$$

From Lemma 2.3,

$$K = \sup u \leq \frac{K + |\gamma|}{2} + \lambda K H_a(|\tilde{\Omega}|).$$

Hence,

$$|\tilde{\Omega}| \geq H_a^{-1} \left( \frac{K - |\gamma|}{2\lambda K} \right).$$

Therefore, from equation (7), we obtain

$$\|w\|_p^p \geq \left( \frac{K + |\gamma|}{2} \right)^p H_a^{-1} \left( \frac{K - |\gamma|}{2\lambda K} \right). \quad (8)$$

Applying  $H_a$  in the inequality, the proof ends.  $\square$

**Remark:** For the case  $\gamma = 0$ , the result is true for any  $\lambda > 0$ , because in this case we may always assume  $\max w = \max |w|$ .

The next lemma is some uniqueness result based on the one established by Brezis and Oswald [2].

**Lemma 2.5.** *Let  $\Omega \subset M$  be a domain with finite measure, possibly unbounded. If  $u_1, u_2 \in H^1(\Omega) \cap C^2(\Omega) \cap C^0(\overline{\Omega})$  satisfying  $0 < u_1 \leq u_2$  are classical solutions of*

$$\begin{cases} -\Delta u &= au + b & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

*for  $a, b$  positive constants, then  $u_1 = u_2$ .*

*Proof.* Since  $u_1, u_2 \in H^1(\Omega) \cap C^2(\Omega) \cap C^0(\overline{\Omega})$  and  $u_1 = u_2 = 0$  on  $\partial\Omega$ , we can prove that  $u_1, u_2 \in H_0^1(\Omega)$ . Hence, from the definition of weak solution,

$$\int_{\Omega} \nabla u_1 \nabla u_2 \, dx = \int_{\Omega} (au_1 + b)u_2 \, dx \quad \text{and} \quad \int_{\Omega} \nabla u_2 \nabla u_1 \, dx = \int_{\Omega} (au_2 + b)u_1 \, dx.$$

Thus

$$\int_{\Omega} \left( \frac{au_1 + b}{u_1} - \frac{au_2 + b}{u_2} \right) u_1 u_2 \, dx = 0.$$

It is then clear that the integrand is nonnegative, which implies that it is equal to zero. Hence

$$\frac{au_1 + b}{u_1} = \frac{au_2 + b}{u_2},$$

and, therefore,  $u_1 = u_2$  completing the proof.  $\square$

**Remark 2.6.** *Lemma 2.5 holds in a more general setting: if  $\beta \geq 0$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz function such that  $f(t + \beta)/t$  is decreasing, then the same conclusion holds considering the problem*

$$\begin{cases} -\Delta u &= f(u) & \text{in } \Omega \\ u &= \beta & \text{on } \partial\Omega. \end{cases}$$

**Lemma 2.7.** *Let  $\lambda > 0$  and  $u \in C^2(\Omega)$  be a positive function that satisfies  $\Delta u + \lambda u \leq 0$  in some domain  $\Omega$  (possibly unbounded) of a Riemannian manifold  $M$ . Then*

$$\lambda \leq \lambda_1(\Omega).$$

*Proof.* Suppose that  $\lambda > \lambda_1(\Omega)$ . There exists a bounded smooth domain  $\Omega_0 \subset\subset \Omega$  with

$$\lambda > \lambda_1(\Omega_0) > \lambda_1(\Omega).$$

Let  $u_0$  be a positive eigenfunction associated to  $\lambda_1(\Omega_0)$  in  $\Omega_0$ . Since  $\Omega_0$  is smooth,  $u_0 \in C^0(\overline{\Omega}) \cap C^2(\Omega_0)$  and  $u_0 = 0$  on  $\partial\Omega_0$ . Since  $u \in C^2(\Omega)$ ,  $u \in C^2(\overline{\Omega_0})$ . Therefore, the positivity of  $u$  in  $\Omega$  implies that  $u \geq c$  in  $\Omega_0$  for some positive constant  $c$ . Taking

$$\alpha := \sup_{\Omega_0} \frac{u_0}{u},$$

that is finite and positive, we have that  $\alpha u - u_0 \geq 0$  in  $\Omega_0$  and  $\alpha u(p) - u_0(p) = 0$  for some point  $p \in \overline{\Omega_0}$ . Observe that  $p \notin \partial\Omega_0$  since  $u_0 = 0$  on  $\partial\Omega_0$ . That is,  $\alpha u - u_0 \geq 0$  has a point of minimum at  $p \in \Omega_0$ . But this contradicts the maximum principle, since

$$-\Delta(\alpha u - u_0) \geq \alpha \lambda u - \lambda_1(\Omega_0)u_0 \geq \alpha \lambda u - \lambda u_0 = \lambda(\alpha u - u_0) \geq 0,$$

proving that  $\lambda \leq \lambda_1(\Omega)$ . □

## 2.2 Proof of Theorem 2.2

*Proof.* We assume that  $w > 0$  in  $U$ , otherwise we split  $U$  into two unbounded domains. Fix a point  $o \in U$  and let  $\gamma = w(o)/2$ . Then the set  $\Omega = \{x \in U : w(x) > \gamma\}$  is not empty and it has finite measure since

$$\gamma^p |\Omega| \leq \int_{\Omega} |w|^p dx < \infty.$$

Moreover, since  $\Delta w + \lambda w = 0$  and  $w$  is positive in  $\Omega$ , it follows from Lemma 2.1 that  $\lambda \leq \lambda_1(\Omega)$ . Now define  $\Omega_k = \Omega \cap B_k(o)$  for  $k \in \mathbb{N}$  and let  $z_k \in H_0^1(\Omega_k)$  be the weak solution of

$$\begin{cases} -\Delta v &= \lambda v + \lambda \gamma & \text{in } \Omega_k \\ v &= 0 & \text{on } \partial\Omega_k \end{cases} \quad (9)$$

If  $\Omega_k = \Omega$ ,  $z_k = w - \gamma$ . Otherwise, the existence of solution to this problem is a consequence of  $\lambda \leq \lambda_1(\Omega) < \lambda(\Omega_k)$  and the classical theory for eigenvalue problems in PDE. Observe that  $w_k := z_k + \gamma$  is a weak solution of

$$\begin{cases} -\Delta v &= \lambda v & \text{in } \Omega_k \\ v &= \gamma & \text{on } \partial\Omega_k \end{cases}$$

and that  $w_k = \gamma \leq w$  on  $\partial\Omega_k$ . Then  $w_k < w$  in  $\Omega_k$  since  $\lambda \leq \lambda_1(\Omega) \leq \lambda_1(\Omega_k)$ . The equality  $\lambda_1(\Omega) = \lambda_1(\Omega_k)$  only happens if  $\Omega = \Omega_k$ . By the same argument,  $w_k \geq w_m > \gamma$  for any  $k > m$ . Hence from (8), for each  $k \in \mathbb{N}$ ,

$$\left(\frac{\|w_k\|_\infty + \gamma}{2}\right)^p H_a^{-1}\left(\frac{\|w_k\|_\infty - \gamma}{2\lambda\|w_k\|_\infty}\right) \leq \|w_k\|_p^p \leq \|w\|_p^p. \quad (10)$$

Since  $H_a^{-1}$  is an increasing positive function, the left-hand side diverges to infinity if  $\|w_k\|_\infty$  goes to infinity. Hence  $(\|w_k\|_\infty)$  is a bounded sequence and, since it is also increasing, converges pointwise to some bounded function  $\bar{w}$  defined on  $\Omega$ . Moreover  $w_k \leq \bar{w} \leq w$ . To prove that  $w$  is bounded, it is sufficient to show that  $w = \bar{w}$  in  $H^1(\Omega)$ . We observe that since  $\Omega$  has finite measure and  $w \in W^{1,p}(\Omega)$  for  $p \geq 2$ ,  $w \in H^1(\Omega)$ .

Let  $z = w - \gamma$  and  $\bar{z} = \bar{w} - \gamma$ . Then  $z_k \leq \bar{z} \leq z$  and  $z_k$  converges pointwise to  $\bar{z}$ . We have to show that  $\bar{z} = z$ . The idea is to show that  $z, \bar{z} \in H^1(\Omega) \cap C^2(\Omega) \cap C^0(\bar{\Omega})$  and satisfy the same Dirichlet problem.

First, notice that  $z \in C^2(\bar{\Omega})$  since  $z = w - \gamma$ ,  $w \in C^2(U)$  and  $\bar{\Omega} \subset U$ . Besides  $z \in H^1(\Omega)$  because  $0 \leq z = w - \gamma \leq w$  and  $w \in H^1(U)$ . Moreover  $z$  is a solution of

$$\begin{cases} -\Delta v &= \lambda v + \lambda \gamma & \text{in } \Omega \\ v &= 0 & \text{on } \partial\Omega. \end{cases} \quad (11)$$

We prove now the similar results for  $\bar{z}$ . Since  $z_k$  is solution of (9),  $z_k \leq w_k \leq w$  and  $\Omega_k \subset \Omega$ , it follows that

$$\int_{\Omega_k} |\nabla z_k|^2 dx = \lambda \int_{\Omega_k} z_k^2 dx + \lambda \gamma \int_{\Omega_k} z_k dx \leq \lambda \int_{\Omega} w^2 dx + \lambda \gamma \int_{\Omega} w dx < \infty.$$

Hence the sequence  $(\|\nabla z_k\|_2)$  is bounded. Moreover,  $z_k \leq w$  implies that  $(\|z_k\|_2)$  is bounded. Therefore, up to some subsequence,  $z_k$  converges weakly to some function in  $H_0^1(\Omega)$ . This limit is  $\bar{z}$ , since  $z_k$  converges pointwise to  $\bar{z}$ . Thus  $\bar{z}$  is a weak solution of (11). Then  $\bar{z}$  is a classical solution and it is of class  $C^2(\Omega)$ . The continuity of  $\bar{z}$  on  $\partial\Omega$  is a consequence of  $0 \leq \bar{z} \leq z$  and of the fact that  $z$  vanishes continuously on  $\partial\Omega$ .

Therefore,  $z$  and  $\bar{z}$  satisfy the hypotheses of Lemma 2.5, which implies uniqueness of solution of problem (11). Hence  $z = \bar{z}$  and  $w = \bar{w}$  in  $\Omega$ , proving that  $w$  is bounded and  $\|w\|_{L^\infty(\Omega)} = \|\bar{w}\|_{L^\infty(\Omega)}$ .

Furthermore, since  $w_k \rightarrow \bar{w}$  and  $w_k \leq \bar{w}$ , we have

$$\|w_k\|_{L^\infty(\Omega)} \rightarrow \|\bar{w}\|_{L^\infty(\Omega)} = \|w\|_{L^\infty(\Omega)}.$$



This implies that (10) also holds replacing  $w_k$  by  $w$ . Observe also that  $\gamma > 0$  can be chosen so small as we want since  $\gamma = w(o)/2$  and  $w$  goes to zero as we approach the boundary. So we can omit  $\gamma$  in (10), obtaining

$$\left(\frac{\|w\|_\infty}{2}\right)^p H_a^{-1}\left(\frac{1}{2\lambda}\right) \leq \|w\|_{L^p(\Omega)}^p,$$

completing the proof.  $\square$

### 3 Application to Hadamard manifolds

Hadamard manifolds admit some isoperimetric function with well-defined a.i.f. and, therefore, Theorem 2.2 applies to them. Indeed it is a consequence of Theorem 3.1 proved by Christopher B. Croke [5] for  $n \geq 3$ . For  $n = 2$  the theorem also holds even for more general manifolds according to E. F. Beckenbach and T. Radó [3].

**Theorem 3.1.** *Let  $N^n$  be a compact Riemannian manifold (with boundary) of nonpositive sectional curvature. Suppose that any geodesic ray in  $N$  minimizes length up to the point it hits the boundary. Then there exists a positive constant  $D(n)$ , that depends only on  $n$ , such that*

$$\text{Vol}(\partial N) \geq D(n)(\text{Vol}(N))^{1-1/n}.$$

If  $M$  is a Hadamard manifold, any smooth compact subdomain satisfies the hypotheses of the theorem. Hence  $H(s) = D(n)s^{1-1/n}$  is an isoperimetric function on  $M$  and

$$H_a(t) = \int_0^t \frac{s}{H(s)^2} ds = \frac{n}{2(D(n))^2} t^{2/n}.$$

**Corollary 3.2.** *Suppose that  $M$  is a Hadamard manifold,  $U$  is an unbounded domain of  $M$  and  $w$  is an eigenfunction (solution of (4)) in  $U$  associated to  $\lambda > 0$ . If  $w \in L^p(U)$  for some  $p \geq 2$ , then  $w$  is bounded and*

$$\|w\|_{L^\infty(U)} \leq \frac{2(n\lambda)^{n/4}}{(D(n))^{n/2}} \|w\|_{L^p(U)}.$$

As an application, we obtain that the eigenfunctions below are bounded.

**Theorem 3.3** (Theorem 1.1 of [6]). *Let  $M$  be a complete simply connected negatively curved Riemannian manifold. If  $\lim K(r) = -\infty$  as  $r \rightarrow \infty$ , then  $\Delta$  has pure point spectrum.*

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